Problem 1: Microscopic Momentum Balance

A rectangular block of wood of size \(2\lambda\) (where \(2\lambda < 1\)) is present in a rectangular channel of width 1 unit. The length of the block is denoted as \(L\) and the width of the block is the same as that of the channel and is denoted as \(W\) (see Figure 1). The viscosity of the fluid is \(\mu\).

When the pressure at the plane 1, \(p_1\), exceeds that at plane 2, \(p_2\), the block starts moving with a constant velocity \(V\).

(a) Solve for the velocity profile in the gap between the block and the channel (you can choose either the top or the bottom).

(b) By using a force balance on the wooden block, derive an expression for the steady state velocity \(V\) in terms of \(p_1, p_2, L, \lambda, \mu\) (note that the pressures \(p_1\) and \(p_2\) also act on the block).

Figure 1
Problem 2: Macroscopic Mass Balance

Hemodialysis or artificial kidney units are used to purify the blood of patients whose natural kidneys are unable to effectively separate undesirable compounds (e.g., urea, uric acid or creatinine) from the blood. The simplest artificial kidney system consists of two compartments (one containing the incoming blood and another – the dialysate – containing a dilute solution of electrolytes in water). The two compartments are separated by a semipermeable membrane which is capable of removing urea, uric acid and creatinine against an isotonic dialysate solution containing electrolytes and various sugars. The purpose of the electrolytes and sugars is to “balance” the same compounds in the blood and not allow their removal from blood.

Consider a simple dialysis unit with countercurrent flow geometry as shown in Figure 2. Now, you are asked to apply principles of mass transfer to calculate a few important parameters.

![Figure 2](image)

This unit is characterized by a constant blood flow rate \(Q_B\) (ml/min) and a constant dialysate flow rate \(Q_D\) (ml/min). The term dialysance \(D_B\) (ml/min) is given by the equation (1) and defines how efficient the process is.

\[
D_B = \frac{N}{C_{B_i} - C_{D_i}} \tag{1}
\]

Here \(N\) (moles/min) is the rate of undesirable solute removed (urea, uric acid or creatinine), and the two concentrations \(C_{B_i}\) and \(C_{D_i}\) are the inlet concentrations (moles/ml) of this solute (urea, uric acid or creatinine) in the blood and dialysate streams, respectively. Finally, the two concentrations \(C_{B_o}\) and \(C_{D_o}\) are the outlet concentrations (moles/ml) of this solute (urea, uric acid or creatinine) in the blood and dialysate streams, respectively.

It is also known that the rate of solute removal \(N\) can be expressed by equation (2)

\[
N = K_O A \bar{\Delta C} \tag{2}
\]

where \(K_O\) is the overall mass transfer coefficient (cm/min), \(A\) (cm\(^2\)) is the area of the membrane and \(\bar{\Delta C}\) is an appropriate mean concentration driving force for the flow geometry, which is based on the inlet and outlet concentrations.

(continued on the next page)
(a) Derive an equation giving the solute removal $N$ as a function of the *four concentrations* noted above and the terms $K_O$ and $A$.

(b) Derive an expression for the dialysance $D_R$ as a function of blood flow rate $Q_B$ and the terms $Z = Q_B / Q_D$ and $X = K_O A / Q_R$.

(c) Calculate the fractional removal of urea, $E = D_R / Q_B$ from the bloodstream per pass. 

**Remember** that for the membrane

$$\frac{1}{K_O} = \frac{1}{k_d} + \frac{1}{P_M} + \frac{1}{k_b}$$  \hspace{1cm} (3)

where $P_M$ is the membrane permeability, $k_d$ is the local mass transfer coefficient on the dialysate side of the membrane, Sh is the Sherwood number, and $k_b$ is the local mass transfer coefficient on the blood side of the membrane.

Here you are given the following characteristics of the artificial kidney:

$Q_D = 30$ ml/min, $Q_B = 200$ ml/min, $A = 1.0$ m$^2$, 
$1/P_M = 15$ min/cm, $1/k_d = 15$ min/cm and $Sh = k_b h / D = 4$. 

Also, $D = 0.8 \times 10^{-5}$ cm$^2$/sec is the diffusion coefficient of urea in the blood, and $h = 4 \times 10^{-2}$ cm is the blood channel height.

(d) If the area is increased to $A = 3.0$ m$^2$ what is the new fractional urea removal per pass? What are the potential advantages and/or problems associated with such a greatly enlarged dialyzer? Comment.
Problem 3: Microscopic Heat Balance

Consider a set of symmetric semi-infinite slabs (thermally bonded) as depicted in Figure 3. We wish to calculate the steady-state heat flux and temperature profile on the right side of the center-line (or center-plane) of the object (i.e., \(0 \leq x \leq x_{III}\); for all three regions).

In region II, for which \(x_I \leq x \leq x_{II}\), the heat generation is non-uniform and described by the function \(S_{II}^o (x-x_I)\) [the heat generation in region II on the other side of the center-line (i.e., \(-x_{II} \leq x \leq -x_I\)) is equivalent and symmetric about the center-line]. Regions I and III have no heat generation. The thermal conductivity of each of the three regions is \(k_I\), \(k_{II}\), and \(k_{III}\), respectively. Note that the temperature at \(x = \pm x_{III}\) is \(T_s\).

(a) Calculate the heat flux in each region as a function of \(x\) and also calculate the temperature in each region as a function of \(x\). You need only do this for \(0 \leq x \leq x_{III}\) since the object is symmetric about \(x = 0\).

![Figure 3](image)
Problem 4: Microscopic Mass Balance

A simplified model for a catalytic reactor is shown in Figure 4.

![Figure 4](image)

An irreversible reaction \(2A \rightarrow B\) occurs on the surface of the spherical catalyst particles. The rate at which component \(A\) is consumed at the particle surface, \(N_{Ax}\), is proportional to the concentration of \(A\) in the gas phase just next to the particle surface, \(C_A(z = \delta)\), as shown here:

\[ N_{Ax} = kC_A(z = \delta). \]

The rate constant for pseudo-first-order surface reaction is \(k\), and \(z\) is the distance away from the catalyst particle surface. Each particle is surrounded by a very thin, stagnant gas layer (i.e., boundary layer) of thickness \(\delta\). At distances further away from the particle surface than \(\delta\), both \(A\) and \(B\) are well mixed by the turbulent flow of gas around the particles. In the boundary layer, \(A\) diffuses to the particle surface, undergoes reaction to \(B\), and \(B\) diffuses out of the boundary layer and into the bulk turbulent flow of gas around the particles. The gas is isothermal, may be modeled as ideal, and the boundary layer thickness, \(\delta\), is much less than the particle radius.

(a) Model the steady-state transport of gas in the boundary layer to derive an analytical expression for the local rate of reaction of \(A\) per unit area of catalyst particles (mols \(A\) consumed/(m\(^2\) s)), \(N_{Ax}\). The mol fraction of \(A\) at the outer edge of the boundary layer (i.e., at \(z=0\)) is \(x_{A0}\).

(b) Simplify \(N_{Ax}\) for the case where the rate of reaction of \(A\) approaches infinity (i.e., the transport through the boundary layer becomes completely diffusion controlled).
Problem 5: Macroscopic Heat Balance

A mountain cabin, built on a slope, has an exposed water pipe that brings water from a nearby lake into the cabin. In the winter, the cabin owner wants to prevent the water in the pipe from freezing, since if stagnant water in the pipe were to freeze, the pipe would rupture due to the expansion of the water upon freezing.

The cabin owner plans to prevent the pipe from rupturing by leaving some of the cabin’s water faucets dripping. This small flow will provide enough additional heat to the pipe to prevent the water in it from freezing.

(a) Determine how much water must flow through the pipe to prevent freezing. Express your answer as gallons per minute.

Make the following assumptions in your analysis:

- The configuration of the pipe is as shown in Figure 5.
- Water enters the pipe from the lake at 4°C and freezing is prevented if water enters the cabin at 2°C.
- The cabin owner wants to prevent freezing even if outside temperatures reach -25°C with winds of 50 km/hr.
- The exposed pipe length is 2 m, the outer diameter of the pipe is 5 cm, pipe walls are then and the radial temperature gradient is negligible (bulk water temperature = outer pipe wall temperature).
- The Nusselt number for heat transfer in this problem (cross-flow for a pipe), where \( \text{Nu} = \text{hD}_o / \text{k}_{\text{air}} \), is given by \( 0.25 \text{ Re}^{0.6} \text{ Pr}^{0.4} \), where \( \text{Pr} = \text{c}_p \mu / k \). \( \text{Pr} \) can be assumed to equal 0.8 in this problem. The Nusselt number is based on heat transfer through the entire external surface of the pipe.
- Axial heat transfer from the pipe to the cabin is negligible, and the difference in temperature driving force along the length of the pipe can be ignored.
- The thermal conductivity, viscosity and density of air are \( 0.022 \text{ W/(m-K)} \), \( 1.3 \times 10^{-5} \text{ kg/(m-s)} \) and \( 1.4 \text{ kg/m}^3 \).
- The heat capacity and density of water are \( 4.2 \text{ kJ/(kg-°C)} \) and \( 1000 \text{ kg/m}^3 \); 1 m\(^3\) = 264 gal.